THE STRUCTURE OF ERGODIC MEASURES FOR COMPACT GROUP EXTENSIONS

BY

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ABSTRACT

In this paper, the structure of the set of invariant measures on a transformation group which is a free compact abelian group extension of another transformation group is studied from both the geometric and analytic viewpoints. It is shown in general that genuine ergodic decompositions are obtained in the non-metric setting for measures that project onto an ergodic measure. In addition, when all the spaces involved are metric, there is a structure theorem for all ergodic measures in terms of the ergodic measures on the base and naturally defined subgroups.

Introduction

In this paper, we study the collection of invariant measures of a transformation group (X, T) which is a free (compact group) G-extension of a transformation group (Y, T). Our purpose is twofold:

(i) to study the structure of invariant measures in the general setting of a compact Hausdorff space X, an arbitrary compact group G, and the requirement that (X, T) admit an invariant measure; and

(ii) a detailed analysis of ergodic measures in the case when X is compact metric, G is compact abelian and T is locally compact and separable.

One problem that arises in studying invariant measures in the non-metric case is the fact that Choquet theory does not necessarily lead to a genuine ergodic decomposition of an invariant measure. In Section 2 we show that we do obtain genuine ergodic decompositions if we restrict our attention to certain classes of invariant measures which arise quite naturally, namely those which project onto ergodic measures (Y, T).

In Section 3 we make the restrictions mentioned in (ii) above, and prove that the ergodic measures on (X, T) which project to a particular ergodic measure on (Y, T) have a rather nice structure. In particular we construct a collection of strictly T-invariant Borel sets each of which supports exactly one of these

Received December 12, 1973 and in revised form April 14, 1974

ergodic measures, and we give a complete description of these measures based on the given ergodic measure on (Y, T) and a certain subgroup of G. Finally we indicate how far these results carry over for a non-metric space X.

In Section 4 we look at general inverse limit systems of transformation groups and show that the corresponding spaces of invariant measures also form an inverse limit system. In particular, if we have an inverse limit sequence (possibly transfinite) of free G-extensions then we can in some sense describe the ergodic measures on the inverse limit. This result may have some interest in the case of a distal flow built by group extensions and inverse limits.

In Section 5 we analyse an example of Furstenberg and show how, by altering the action, we can obtain all the different types of ergodic extensions for circle extensions. We also generalise this slightly to groups with monothetic subgroups. Finally, we make some remarks about entropy in the case of integer actions.

In Section 6 we briefly indicate similar results which can be obtained for minimal sets in free topologically simple G-extensions. The only restriction here is that G be abelian.

At various points in the work we have tried to indicate the roles played by our different assumptions on X, G and T, and to point out some problems which arise in these connections. We also indicate how far our results apply to homomorphisms more general than G-extensions.

Finally, we wish to emphasize in the case of an integer action (i.e., a homeomorphism) on a compact metric space, some of the results here can be obtained in a relatively simple and straightforward fashion. The basic reason for this is the existence of ergodic sets and quasi-regular points. The thrust of this study is to show that in the more general setting with regard to both the space and the acting group, one can obtain the corresponding results and extensions by avoiding the ergodic theorem and using more careful analysis and geometry. In some sense, we feel that these methods and the application to the objects described in this paper are an indication of the proper setting in which to study these systems.

Some of these results have been announced in [8].

1. Preliminaries

1.1. Notation and assumptions. A measure is to be regarded as a regular Borel probability measure, with one exception described in (1.2), and other exceptions in the general discussion of Choquet theory in Section 2. If μ is a measure we use the notation $[\mu]$ to mean "almost everywhere relative to μ".

Maps are to be regarded as continuous unless otherwise specified.

We let (X, T) denote a transformation group, where X is a compact Hausdorff space and T is an arbitrary topological group. The notation $(t \in T)$ at the end of a statement means that the statement holds for all $t \in T$. We let $\mathscr{B}(X)$ denote the collection of Borel sets of X and $M(X, T)$ the collection of T-invariant measures on X. We shall always assume that $M(X, T)$ is non-empty; various assumptions on X and T which arise frequently will imply this fact. For example if T is discretely amenable (see (2.3) for a definition, which includes T abelian), or if (X, T) is a distal flow [6], [4, Prop. 4.3, p. 164], then $M(X, T)$ is non-empty.

We let G denote a compact group. In some sections, we make extra hypotheses on G which will stand throughout that section. If G is abelian then we denote its character group by $\chi(G)$. We use the symbol K for the circle group of complex numbers of absolute value 1 and $\mathscr{B}(X, K)$ for the collection of Borel maps from X to K.

1.2 Measures on transformation groups. A measure $\mu \in M(X, T)$ is called *ergodic* if $A \in \mathscr{B}(X)$ and $\mu(tA \,\Delta\, A) = 0$ $(t \in T)$, implies $\mu(A) = 0$ or 1. The collection of ergodic measures will be denoted by $E(X, T)$. With the weak topology, $M(X, T)$ is a compact convex set and $E(X, T)$ is its set of extreme points. Under our assumption that $M(X, T)$ is non-empty it follows that $E(X, T)$ is non-empty. For a general reference, see [11, Sect. 10].

By a *homomorphism* $\Phi: (X, T) \to (Y, T)$ we mean an onto map $\Phi: X \to Y$ such that $\Phi(tx) = t\Phi(x)$. If $\Phi: (X, T) \to (Y, T)$ is a homomorphism then there is an induced map $\Phi^*: M(X, T) \to M(Y, T)$, which is continuous in the weak topologies, given by $\Phi^*\mu(f) = \mu(f \circ \Phi)$ $(f \in \mathscr{C}(Y),\ \mu \in M(X, T))$. Here we use the notation $\mu(f)$ for $\int f d\mu$. If $m \in M(Y, T)$ then we denote by $\Phi^{-1}m$ the measure (*not* regular Borel) which is defined on the σ-algebra $\Phi^{-1}\mathscr{B}(Y)$ by the relation $\Phi^{-1}m(\Phi^{-1}B) = m(B)$ $(B \in \mathscr{B}(Y))$. Note if $\Phi^*\mu = m$ then μ and $\Phi^{-1}m$ agree on the σ-algebra $\Phi^{-1}\mathscr{B}(Y)$. This fairly trivial observation has some important consequences in formulating our results.

1.3 G-extensions. Let (X, T) be a transformation group and let G act freely on X in such a way that the actions of G and T commute. That is, there is a map $G \times X \to X$, $(g, x) \to gx$, such that:

(i) $g_1(gx) = (g_1 g)x$ $(g_1, g \in G)$;

(ii) $gx = x$ if and only if $g = e$, e denoting the identity element of G;

(iii) $t(gx) = g(tx)$ $(g \in G,\ t \in T,\ x \in X)$.

It follows that for each $g \in G$ the map $x \to gx$ is an isomorphism of (X, T) to itself. Requirement (iii) implies that there is induced an action of T on the orbit space X/G given by $t(Gx) = G(tx)$. We say that (X, T) is a *free G-extension* of (Y, T) if (Y, T) is isomorphic to $(X/G, T)$. When this is the case we will regard (Y, T) as identified with $(X/G, T)$ and $\pi:(X, T)\to(Y, T)$ will denote the map which sends a point onto its G-orbit. Should we wish to emphasize G, we will write $\pi:(G; X, T)\to(Y, T)$, and we will also often refer to π as a free G-extension.

If H is a closed subgroup of G then we can "split" the free G-extension $\pi:(G; X, T)\to(Y, T)$ into a free H-extension $\pi_1:(H; X, T)\to(X/H, T)$ and a homomorphism $\pi_2:(X/H, T)\to(Y, T)$, where π_2 is defined by the requirement $\pi_2\circ\pi_1 = \pi$. If H is a normal subgroup of G then there is a natural action of the quotient group G/H on X/H defined by $(gH)\ (Hx) = H(gx)\ (gH \in G/H, Hx \in X/H)$. In this case π_2 becomes a free G/H-extension, $\pi_2:(G/H; X/H, T)\to(Y, T)$. This splitting is of particular importance when we study the ergodic measures on abelian G-extensions. In fact to each $m \in E(Y, T)$, we will associate a subgroup $H(\pi, m)$ of G, and then all the measures $\nu \in E(X, T)$ with $\pi^*\nu = m$ can be described by two simple operations over the splitting by $H(\pi, m)$.

1.4 Functions of type γ. Throughout this subsection we assume that G is abelian and that (X, T) is a free G-extension of (Y, T).

DEFINITION 1.4.1. Let $\gamma \in \chi(G)$ .An $f \in \mathcal{B}(X, K)$ is said to be of *type* γ if $f(gx) = \gamma(g)f(x)$ $(x \in X, g \in G)$. We reserve the notation f_γ for a function of type γ.

Parry [8, p. 100] has shown that for each $\gamma \in \chi(G)$ there exist functions of type γ. His result is stated for compact metric Y but the proof only uses the compactness of Y and the normality of X and so applies to the situation that we are studying.

Now fix a measure $m \in E(Y, T)$ and let $\mu \in M(X, T)$ satisfy $\pi^*\mu = m$. We are interested in the following equations for functions of type γ:

(1) $f_\gamma \circ t = f_\gamma[\mu]$ $(t \in T)$,

(2) $\dfrac{f\circ\pi\circ t}{f\circ\pi} = \dfrac{f_\gamma\circ t}{f_\gamma}$ $[\mu]$ $(t \in T)$ for some $f \in \mathcal{B}(Y, K)$.

The first observation we make is that if either of these two equations is satisfied for an $x \in X$ then it is satisfied for all gx in the G-orbit of x. In other words, we are only measuring collections of G-orbits and we claim that we can replace μ by $\pi^{-1}m$ in (1) and (2) (see (1.2) for the definition of $\pi^{-1}m$). This means that the sets of γ's for which these equations have solutions depend only on the extension π and the measure m. To show our claim, consider $A = \{x : f_\gamma(tx) = f_\gamma(x) \ (t \in T)\}$. Then $A = \pi^{-1}\pi A$, and moreover, $\mu(A) = 1$. Now let (A_n) be a sequence of compact subsets of A with $\mu(A_n) \to 1$. Since for each n, $\pi(A_n) \in \mathcal{B}(Y)$, then $\bigcup_n \pi(A_n) \in \mathcal{B}(Y)$ and $m(\bigcup_n \pi(A_n)) = 1$. Hence, if $A_0 = \pi^{-1}(\bigcup_n \pi(A_n))$, then $A_0 \in \pi^{-1}(\mathcal{B}(Y))$, and $\pi^{-1}m(A_0) = 1$. Restricting our equations to A_0 gives the desired result. A similar argument holds for the second equation.

The second observation is that if equation (1) is satisfied for some f_γ, then equation (2) is satisfied with $f \equiv 1$. Conversely, if equation (2) is satisfied for some f_γ and f, then (1) is satisfied with the function $f_\gamma/(f \circ \pi)$ which is also of type γ. Using these observations, we now define a subset of $\chi(G)$.

Definition 1.4.2. We denote by $\Gamma(\pi, m)$ the set of $\gamma \in \chi(G)$ for which there is a function f_γ of type γ satisfying $f_\gamma \circ t = f_\gamma [\pi^{-1}m] \ (t \in T)$. Alternatively, $\Gamma(\pi, m)$ is the set of $\gamma \in \chi(G)$ for which there is a function f_γ of type γ and a function $f \in \mathcal{B}(Y, K)$ satisfying

$$\frac{f \circ \pi \circ t}{f \circ \pi} = \frac{f_\gamma \circ t}{f_\gamma} \ [\pi^{-1}m] \quad (t \in T).$$

The advantage of the second description is that if one function of type γ satisfies the above equation then every function of type γ also satisfies it (with different $f \in \mathcal{B}(Y, K)$). See [9, p. 98].

Note that $\Gamma(\pi, m)$ is a subgroup of $\chi(G)$. We will denote by $H(\pi, m)$ the subgroup of G which annihilates $\Gamma(\pi, m)$.

1.5. Convolutions of measures. Let $\pi : (G; X, T) \to (Y, T)$ and denote by $P(G)$ the collection of regular Borel probability measures on G. We will define a convolution operation of $P(G)$ on $M(X, T)$ and discuss its elementary properties. We use the same notation for this operation and the usual convolution operation between elements of $P(G)$.

Definition 1.5.1. For $\alpha \in P(G)$, $\mu \in M(X, T)$ we define $\alpha * \mu$ by

$$\alpha * \mu(f) = \int_G \int_X f(gx) d\mu(x) d\alpha(g) \qquad (f \in \mathcal{C}(X)).$$

Proposition 1.5.2. (1) $\alpha * \mu \in M(X, T) \quad (\alpha \in P(G), \ \mu \in M(X, T))$.
(2) $\alpha * (\beta * \mu) = (\alpha * \beta) * \mu \quad (\alpha, \beta \in P(G), \ \mu \in M(X, T))$.

(3) $(a_1\alpha_1 + a_2\alpha_2) * \mu = a_1(\alpha_1 * \mu) + a_2(\alpha_2 * \mu)$ $(\alpha_1, \alpha_2 \in P(G),\ \mu \in M(X,T))$ *where* $0 \leq a_i \leq 1$, $i = 1,2$, *and* $a_1 + a_2 = 1$.

(4) *The operation* $*$ *from* $P(G) \times M(X,T)$ *to* $M(X,T)$ *is continuous in the weak topologies.*

PROOF. These properties are easy consequences of the definitions.

The set $P(G)$ is a topological semigroup under the convolution operation and parts (1), (2), and (4) of Proposition 1.5.2 imply that we have a continuous semigroup action of $P(G)$ on $M(X,T)$. If we associate to each element $g \in G$ the atomic measure δ_g which gives measure 1 to the element g then we obtain an embedding of G in $P(G)$ and hence an action of G on $M(X,T)$. In accordance with usual practice we will denote $\delta_g * \mu$ by $g\mu$; $g\mu(A) = \mu(g^{-1}A)$. The action of $P(G)$ on $M(X,T)$ is not free and in order to deal with this we will define an equivalence relation on $P(G)$ via a subgroup H of G.

DEFINITION 1.5.3. Let $\pi_H: G \to G/H$ denote the natural map from G to the set G/H of left H-cosets. We say that $\alpha, \beta \in P(G)$ are *equivalent mod H* if the induced measures $\pi_H^*\alpha$ and $\pi_H^*\beta$ coincide on G/H.

If $\alpha \in P(G)$ is such that $\pi_H^*\alpha$ is a point mass measure on G/H then we say that α is a *point mass mod H*.

Note that $\pi_H^*P(G) = P(G/H)$ and the above relation is the one induced by this map. Also, $P(G/H)$ is a compact convex set with the point mass measures as extreme points.

THEOREM 1.5.4. *For* $\mu \in M(X,T)$, *define the subgroup* H_μ *of* G *by* $H_\mu = \{g \in G : g\mu = \mu\}$. *Then the orbit* $P(G) * \mu$ *is the image of* $P(G/H_\mu)$ *under the affine map* $\pi_{H_\mu}^*\alpha \to \alpha * \mu$. *If* $\mu \in E(X,T)$, *then* $P(G) * \mu$ *is affinely homeomorphic to* $P(G/H_\mu)$.

PROOF. For simplicity, we denote H_μ by H and the map by ψ. We first show that ψ is well-defined. Let $\alpha, \beta \in P(G)$ with $\pi_H^*\alpha = \pi_H^*\beta$. Now if $f \in \mathscr{C}(X)$, then the function $g\mu(f)$, as a function of G, is constant on left H-cosets, and thus defines a continuous function on G/H which we denote by $\overline{\mu f}$. Note that $\overline{\mu f}([g]) = g\mu(f)$, if $[g] \in G/H$. We then have

$$\alpha * \mu(f) = \int_G g\mu(f)d\alpha = \int_{G/H} \overline{\mu f}([g])d\pi_H^*\alpha = \int_{G/H} \overline{\mu f}([g])d\pi_H^*\beta$$

$$= \int_G g\mu(f)d\beta = \beta * \mu(f).$$

Since f is arbitrary, $\alpha * \mu = \beta * \mu$.

It is direct to verify that ψ is affine and continuous. Since $\pi_H^*P(G) = P(G/H)$, ψ is onto.

Finally, suppose μ is ergodic. Then $g\mu$ is ergodic ($g \in G$), and $G/H \to E(X,T)$, $[g] \to g\mu$, is a homeomorphism onto. Suppose $\alpha * \mu = \beta * \mu = \nu$. Then

$$\nu(f) = \int_{G/H} g\mu(f)d\pi_H^*\alpha([g]) = \int_{G/H} g\mu(f)d\pi_H^*\beta([g]) \quad (f \in \mathscr{C}(X)),$$

and these give integral representations for ν over the extreme points $E(X,T)$ of $M(X,T)$. Since $M(X,T)$ is a simplex, it follows from general Choquet theory (cf. (2.1)) that these two representations are equal, and hence $\pi_H^*\alpha = \pi_H^*\beta$. Thus, ψ is one-to-one, completing the proof.

Corollary 1.5.5. (1) *The extreme points of $P(G)*\mu$ are those $\alpha * \mu$ for which α is equivalent to a point mass mod H_μ.*

(2) *If μ is ergodic, there exists exactly one G-invariant measure in $P(G)*\mu$, namely $\lambda * \mu$, where λ denotes Haar measure on G.*

Proof. The first assertion follows because the point masses are the extreme points of $P(G/H_\mu)$, and every extreme point of $P(G)*\mu$ is the image under the affine map ψ of an extreme point of $P(G/H_\mu)$.

The second assertion follows because there is a unique "Haar" measure on G/H_μ which is invariant under the natural action of G on G/H_μ, namely $\pi^*_{H_\mu}\lambda$.

Remark. If μ is ergodic, then the extreme points of $P(G)*\mu$ are homeomorphic to G/H_μ and the representation $\alpha * \mu$ may be regarded as an integral representation over the extreme points with representing measure $\pi_H^*\alpha$. Hence, all the extreme points of $P(G)*\mu$ are ergodic, and we obtain ergodic decompositions for the measures $\alpha * \mu$.

Corollary 1.5.6. *If ν is ergodic and $\lambda * \nu$ is ergodic, $P(G)*\nu = \{\nu\}$.*

Proof. If $\lambda * \nu$ is ergodic then it is extreme in $P(G)*\nu$ and hence λ is equivalent to a point mass mod H_ν. This can only happen if $H_\nu = G$. But then $G/H_\nu = \{e\}$ and hence $P(G)*\nu$ consists of the single point $\nu = \lambda * \nu$.

In Section 2 we will show that the sets $P(G)*\mu$, $\mu \in E(X,T)$, occur quite naturally in the study of G-extensions.

1.6. Conditional expectation. At some points in this paper we will have to make use of the conditional expectation operator and its properties (see Billingsley, [1, p. 106ff], for more details).

Let $(X,\mathscr{B},\mu)$ be a probability space, let $f \in L_1(X,\mathscr{B},\mu)$ and let $\mathscr{C}$ be a sub-σ-algebra of $\mathscr{B}$. We define a set function ν on $\mathscr{C}$ by $\nu(C) = \int_C f d\mu, (C \in \mathscr{C})$. It is easy to verify that ν is absolutely continuous with respect to the restriction of μ to $\mathscr{C}$ and hence by the Radon-Nikodym theorem there is an integrable,

$\mathscr{C}$-measurable function $E(f/\mathscr{C})$ which satisfies $\int_C E(f/\mathscr{C})d\mu = \int_C f d\mu$, $(C \in \mathscr{C})$. We call the function $E(f/\mathscr{C})$ the *conditional expectation of f given* $\mathscr{C}$. We list all the properties that we shall need in one theorem.

THEOREM 1.6.1. *Let the notation be as above. Then*:

(1) *$E(f/\mathscr{C})$ is unique $[\mu]$ and is determined by the requirements that it be $\mathscr{C}$-measurable and satisfy*

$$\int_C E(f/\mathscr{C})d\mu = \int_C f d\mu \qquad (C \in \mathscr{C}).$$

(2) *If f is $\mathscr{C}$-measurable and if f and fg belong to $L_1(X, \mathscr{B}, \mu)$ then $E(fg/\mathscr{C}) = f$ $E(g/\mathscr{C})$ $[\mu]$.*

(3) *Let $\{\mathscr{C}_n\}$ be an increasing sequence of sub-σ-algebras with $\mathscr{C}_n \uparrow \mathscr{C}$. Then $\lim_{n\to\infty} E(f/\mathscr{C}_n) = E(f/\mathscr{C})$ $[\mu]$ (Martingale theorem).*

(4) *If S is a measure-preserving transformation from $(X, \mathscr{B}, \mu)$ to itself, then $E(f \circ S/S^{-1}\mathscr{C}) = E(f/\mathscr{C}) \circ S[\mu]$.*

1.7. A lemma on T-invariant sets. Let T be a locally compact separable group and let ξ denote a left invariant Haar measure on T. Let (X, T) be a transformation group and suppose A is a Borel subset of X. We then define the set $A_0 = \{x \in X : tx \in A \text{ for } \xi\text{-almost all } t \in T\}$. It is easily verified that A_0 is a strictly T-invariant subset of X.

For a function $\Phi \in L_1(T, \xi)$ and a bounded Borel function f on X we define a convolution $\Phi * f(x) = \int \Phi(t) f(t^{-1}x)d\xi(t)$. Then $\Phi * f$ is a bounded Borel function on X. Let $S = \{\Phi \in L_1(T, \xi) : \Phi \geq 0, \|\Phi\|_1 = 1\}$. Given a Borel subset A of X, define $A_1 = \{x : \Phi * \chi_A(x) = 1 \text{ for all } \Phi \in S\}$, where χ_A denotes the characteristic function of A.

LEMMA 1.7.1. *The sets A_0 and A_1 coincide.*

PROOF. If $x \in A_0$ then the function $\chi_A(t^{-1}x)$, as a function on T, is equal to 1 $[\xi]$. Therefore if $\Phi \in S$,

$$\Phi * \chi_A(x) = \int_T \Phi(t)\chi_A(t^{-1}x)d\xi(t) = \int_T \Phi(t)d\xi(t) = \|\Phi\|_1 = 1,$$

and so $x \in A_1$.

If $x \notin A_0$ then there is a subset T_0 of T such that $0 < \xi(T_0) < \infty$ and if $t^{-1} \in T_0$ then $t^{-1}x \notin A$. Define $\Phi(t) = (1/\xi(T_0))\chi_{T_0}(t^{-1})$. Then $\Phi \in S$ and $\Phi * \chi_A(x) = 0$. Therefore $x \notin A_1$.

LEMMA 1.7.2. *The set A_0 is Borel. If $\mu \in M(X, T)$ and A satisfies $\mu(A \Delta tA) = 0$ $(t \in T)$, then $\mu(A \Delta A_0) = 0$.*

PROOF. Varadarajan [13, p. 196] proves this for A_1.

2. Choquet theory

The main reference for this section is Phelps [10], particularly Sections 4, 9 and 10. In (2.1), we give a number of definitions and results from this source, and in (2.2), these are applied to the study of a particular class of invariant measures for G-extensions. In (2.3), we indicate how far these results apply to more general homomorphisms.

2.1. General Choquet theory. Suppose F is a non-empty compact convex subset of a locally convex space E and let μ be a probability measure on F. A point $x \in F$ is said to be *represented by* μ if $f(x) = \int_F f d\mu$ for every continuous linear functional f on E. We will say that μ is *concentrated* on a Borel set S of F if $\mu(S) = 1$. It is always the case that a point z in the extreme points, $\mathscr{E}(F)$, of F is uniquely represented by point mass at z. The main purpose of Choquet theory is to try to find representing measures μ which are concentrated on $\mathscr{E}(F)$, and to find conditions for uniqueness of representing measures μ. The possibility of achieving this depends on F being a simplex and μ being a maximal measure. We now give these definitions.

DEFINITION 2.1.1 ([11, p. 59]). A compact convex set F is the *base* of a convex cone $\tilde{F}$ if there is a cone $\tilde{F} \subset E$ with vertex at the origin such that $y \in \tilde{F}$ if and only if there is a unique $a \geq 0$ and $x \in F$ with $y = ax$.

A cone $\tilde{F}$ induces a partial ordering on E by $x \geq y$ if and only if $x - y \in \tilde{F}$.

If F is the base of a cone $\tilde{F}$ then F is called a *simplex* if the subspace $\tilde{F} - \tilde{F} = \{w - z : w, z \in F\}$ generated by $\tilde{F}$ is a lattice in the partial ordering on E induced by $\tilde{F}$. In other words, if $x, y \in \tilde{F} - \tilde{F}$, then the greatest lower bound $x \wedge y$ also belongs to $\tilde{F} - \tilde{F}$.

DEFINITION 2.1.2 [11, p. 24]. If λ and μ are nonnegative Borel probability measures on F we write $\lambda > \mu$ if $\lambda(f) \geq \mu(f)$ for each continuous convex function f on F. A measure μ is called *maximal* if it is maximal with respect to this partial ordering.

The Choquet-Meyer theorem [11, p. 66] states: *If F is a simplex then for each $x \in F$ there is a unique maximal measure μ_x which represents x.*

We shall need a few technical facts about maximal measures.

THEOREM 2.1.3 [11, p. 30, 70]. (1) *A maximal measure μ gives zero measure to any Baire subset of $F - \mathscr{E}(F)$, and to any G_δ subset of $F - \mathscr{E}(F)$. In particular, a maximal measure is concentrated on the closure of $\mathscr{E}(F)$.*

(2) *If μ is a nonnegative probability measure which vanishes on every compact subset of $F - \mathscr{E}(F)$, then μ is maximal. In particular, if μ is concentrated on a subset of $\mathscr{E}(F)$, then μ is maximal.*

2.2 Choquet theory and G-extensions. Let (X, T) be a free G-extension of (Y, T), and $\pi:(G; X, T) \to (Y, T)$. Recalling the notation of (1.2), it is well known that $M(X, T)$ is compact in the topology of weak convergence and is a simplex ([11, p. 80 ff]). Hence, for each $\mu \in M(X, T)$, there is a unique maximal probability measure β on $M(X, T)$ such that

$$\mu(f) = \int_{M(X,T)} \nu(f) d\beta(\nu) \qquad (f \in \mathscr{C}(X)).$$

REMARK. If X is compact metric, then so is $M(X, T)$, and $E(X, T)$ is a Baire subset. Hence, we can assert that β is concentrated on $E(X, T)$ and obtain a unique "ergodic decomposition" of μ. However, in the case when X is not metric, the general Choquet theory does not allow us to assert that β is concentrated on $E(X, T)$ and so we do *not* obtain an ergodic decomposition of μ.

Our main purpose in this subsection is to show that when (X, T) is a free G-extension of (Y, T) we can consider a certain fairly natural restriction on $\mu \in M(X, T)$ which will then allow us to obtain an *ergodic* decomposition. This restriction is that $\pi^*\mu \in \mathscr{E}(Y, T)$, that is, we fix an ergodic measure m on (Y, T) and then consider the subset $P_m \subset M(X, T)$ defined by $P_m = \{\mu \in M(X, T): \pi^*\mu = m\}$.

REMARK. The set P_m is always non-empty. Given $m \in E(Y, T)$ we define a measure $\tilde{m} \in P_m$ by putting

$$\tilde{m}(f) = \int_X \int_G f(gx) d\lambda(g) d\pi^{-1}m(x) \qquad (f \in \mathscr{C}(X)),$$

where λ denotes normalised Haar measure on G. The inner integral leads to a function which is measurable with respect to $\pi^{-1}\mathscr{B}(Y)$ and so its integral with respect to $\pi^{-1}m$ is meaningful. It is clear that $\pi^*\tilde{m} = m$, and hence $\tilde{m}$ belongs to P_m. We call $\tilde{m}$ the *Haar lift of m*.

As a first step in our analysis of the structure of P_m, we show that it is invariant under the action of $P(G)$.

LEMMA 2.2.1. *If $\mu \in P_m$, then $P(G) * \mu \subset P_m$.*

PROOF. Let $f \in \mathscr{C}(Y)$, $\beta \in P(G)$. Then

$$\begin{aligned}\beta * \mu(f \circ \pi) &= \int_G \int_X f(\pi(gx)) d\mu(x) d\beta(g) \\ &= \int_G \int_X f(\pi(x)) d\mu(x) d\beta(g) = \int_G m(f) d\beta = m(f),\end{aligned}$$

where the second equality follows because $\pi(gx) = \pi(x)$. Therefore, $\pi^*(\beta * \mu) = m$ and so $\beta * \mu \in P_m$.

In order to apply Choquet theory we prove:

THEOREM 2.2.2. *The set P_m is a simplex.*

PROOF. The proof basically copies the corresponding proof for $M(X,T)$ as in [11, p. 78 ff]. First note that since $P_m = (\pi^*)^{-1}\{m\}$, P_m is compact and convex.

Now let $\tilde{P}_m$ be the set of finite T-invariant measures μ on X such that $\pi^*\mu = \mu(X)m$. Then $\tilde{P}_m$ is a cone with P_m as its base. To show that $\tilde{P}_m - \tilde{P}_m$ is a lattice in the space of signed measures, it suffices to show that if $\mu, \nu \in \tilde{P}_m$ then $\mu \wedge \nu \in \tilde{P}_m$. The greatest lower bound $\mu \wedge \nu$ of μ and ν is defined in terms of Radon-Nikodyn derivatives by

$$h(x) = \frac{d\mu \wedge \nu}{d(\mu+\nu)}(x) = \min\left\{\frac{d\mu}{d(\mu+\nu)}(x), \frac{d\nu}{d(\mu+\nu)}(x)\right\}$$

To complete the proof, we need to show that $\pi^*(\mu \wedge \nu) = cm$. Let $f \in \mathscr{C}(Y)$. Then

$$\begin{aligned}\pi^*(\mu\wedge\nu)(f) &= \int_X f(\pi x)h(x)d(\mu+\nu)\\ &= \int_X E(f(\pi x)\cdot h(x)/\pi^{-1}\mathscr{B}(Y))d(\mu+\nu)\\ &= \int_X f(\pi x)E(h/\pi^{-1}\mathscr{B}(Y))d(\mu+\nu)\end{aligned}$$

since $f(\pi x)$ is measurable $\pi^{-1}\mathscr{B}(Y)$ (see (1.5) for the properties of conditional expectations). Since $\mu \wedge \nu$ is T-invariant, we have $h \circ t = h$ $[\mu+\nu]$ $(t \in T)$ and since also $\pi^{-1}\mathscr{B}(Y)$ is T-invariant, we have $E(h/\pi^{-1}\mathscr{B}(Y))\circ t = E(h/\pi^{-1}\mathscr{B}(Y))$ $[\mu+\nu]$ $(t \in T)$. Now $\pi^*(\mu+\nu) = (\mu(X)+\nu(X))\cdot m$ and m is ergodic, so $E(h/\pi^{-1}\mathscr{B}(Y))$ is constant $[\mu+\nu]$. Denoting this constant by a we have

$$\begin{aligned}\pi^*(\mu\wedge\nu)(f) &= \int_X f(\pi x)\cdot a\cdot d(\mu+\nu)\\ &= a\cdot(\mu(X)+\nu(X))\cdot m(f).\end{aligned}$$

The proof is complete.

Since P_m is a simplex we can make use of general Choquet theory in P_m and assert the existence of unique maximal measures representing measures on P_m for points in P_m. We now show that the extreme points in P_m are precisely the ergodic measures in P_m and these form a closed subset. Hence we obtain representing measures for $M(X,T)$ which are concentrated on ergodic measures in P_m and therefore ergodic decompositions for the measures in P_m.

THEOREM 2.2.3. (1) *The intersection $P_m \cap E(X,T)$ is non-empty and is equal to $\mathscr{E}(P_m)$.*

(2) *If* $\nu, \eta \in P_m \cap E(X, T)$, *then there exists a* $g \in G$ *such that* $\nu = g\eta$. *Thus, if* $\nu \in P_m \cap E(X, T)$, *then* $\mathscr{E}(P_m) = P_m \cap E(X, T) = \{g\nu : g \in G\}$.

PROOF. (1) It is clear that $\mathscr{E}(P_m)$ is non-empty and contains $P_m \cap E(X, T)$. We will show $\mathscr{E}(P_m) \subset P_m \cap E(X, T)$.

Let $\mu \in \mathscr{E}(P_m)$ and assume that in $M(X, T)$ we can write $\mu = a_1\mu_1 + a_2\mu_2$, where $a_1, a_2 \geq 0$ and $a_1 + a_2 = 1$. Now apply π^* to this relation to obtain $a_1\pi^*\mu_1 + a_2\pi^*\mu_2 = \pi^*\mu = m$. But m is ergodic and so $\pi^*\mu_1 = \pi^*\mu_2 = m$. Therefore $\mu_1, \mu_2 \in P_m$. Since $\mu \in \mathscr{E}(P_m)$, this means $\mu_1 = \mu_2 = \mu$, and so μ is extreme in $M(X, T)$. That is, $\mu \in E(X, T)$.

(2) Let $\nu, \eta \in P_m \cap E(X, T)$ and assume the sets $\{g\nu : g \in G\}$ and $\{g\eta : g \in G\}$ are disjoint. Consider the measures $\lambda * \nu$ and $\lambda * \eta$. Then

$$\begin{aligned} \lambda * \nu(f) &= \int_G \int_X f(gx)d\eta(x)d\lambda(g) \\ &= \int_X \int_G f(gx)d\lambda(g)d\eta(x) \\ &= \int_X \int_G f(gx)d\lambda(g)d\pi^{-1}m(x) \\ &= \tilde{m}(f). \end{aligned}$$

The third equality follows because the inner integral is measurable $\pi^{-1}\mathscr{B}(Y)$, and η agrees with $\pi^{-1}m$ on $\pi^{-1}\mathscr{B}(Y)$. Similarly $\lambda * \eta = \tilde{m}$.

As pointed out in (1.5), the equation $\tilde{m} = \lambda * \nu$ can be regarded as a representation of $\tilde{m}$ by a probability measure on $\{g\nu : g \in G\} \subset \mathscr{E}(X, T)$ and hence is a representation by a maximal measure (see Theorem 2.1.3). Similarly, the equation $\tilde{m} = \lambda * \eta$ yields a representation by a maximal measure concentrated on $\{g\eta : g \in G\} \subset \mathscr{E}(X, T)$. The disjointness of these subsets would contradict the uniqueness of maximal representing measures. Therefore, there is a $g \in G$ such that $\nu = g\eta$. Finally, the last remark in the theorem summarises the above two parts of the theorem, and this completes the proof.

COROLLARY 2.2.4. *The set* $\mathscr{E}(P_m)$ *is a compact subset of* $M(X, T)$. *Every* $\mu \in P_m$ *can be uniquely represented by a probability measure concentrated on* $\mathscr{E}(P_m)$. *Hence, every* $\mu \in P_m$ *has a unique ergodic decomposition.*

PROOF. Let $\nu \in \mathscr{E}(P_m)$ and let $H_\nu = \{g : g\nu = \nu\}$. Then H_ν is a closed subgroup of G and the set $\{g\nu : g \in G\}$, which is $\mathscr{E}(P_m)$, is homeomorphic to G/H_ν and hence is compact.

Notice that if we choose a different element $\nu_0 \in \mathscr{E}(P_m)$ then we obtain a subgroup H_{ν_0} which is a conjugate of H_ν: if $\nu_0 = g_0\nu$ then $H_{\nu_0} = g_0 H_\nu g_0^{-1}$. The spaces G/H_ν and G/H_{ν_0} are homeomorphic, and no ambiguity occurs.

For the final part of the corollary we use the Choquet-Meyer theorem on the simplex P_m to obtain a maximal representing measure for $\mu \in P_m$. Since $\mathscr{E}(P_m)$ is closed this measure is concentrated on $\mathscr{E}(P_m)$. And from the fact that $\mathscr{E}(P_m) \subset E(X, T)$ it follows that this representing measure is also a maximal measure on $M(X, T)$ and hence gives a unique ergodic decomposition. This completes the proof.

To complete the geometric picture of P_m we combine this section with (1.5) to obtain:

THEOREM 2.2.5. *Choose any $\nu \in \mathscr{E}(P_m)$. Then $P_m = P(G) * \nu$, and is affinely homeomorphic to $P(G/H_\nu)$.*

PROOF. We have shown in Lemma 2.2.1 that $P(G) * \nu \subset P_m$. Now if $\mu \in P_m$, then there exists a measure β concentrated on $\mathscr{E}(P_m)$ such that

$$\mu(f) = \int_{\mathscr{E}(P_m)} \nu'(f)\, d\beta(\nu'), \qquad (f \in \mathscr{C}(X)).$$

Let $\Phi: G/H_\nu \to \mathscr{E}(P_m)$ be the homeomorphism given by $\Phi(gH_\nu) = g\nu$. Let $\beta' = (\Phi^{-1})^*\beta$ and let α be any measure on G such that $\pi^*_{H_\nu}\alpha = \beta'$, where $\pi_{H_\nu}: G \to G/H_\nu$ is the canonical map. Then since $g\nu(f)$ is constant on H_ν-cosets we have

$$\mu(f) = \int_G g\nu(f) d\alpha(g) = \alpha * \nu(f), \qquad (f \in \mathscr{C}(X)).$$

Therefore $\mu \in P(G) * \nu$. That is, $P_m = P(G) * \nu$. The final statement now follows from Theorem 1.5.4.

COROLLARY 2.2.6. *P_m is a singleton if and only if $\tilde{m} \in E(X, T)$. In this case, $P_m = \{\tilde{m}\}$.*

PROOF. This follows from (1.5.6) and (2.2.5).

This last corollary is a generalization of a result of Parry [9]. Also note that during the proof of Theorem 2.2.5 we have shown that if $\mu = \alpha * \nu$, then the representing measure for μ is $\pi^*_{H_\nu}\alpha$, if we identify G/H_ν and $\mathscr{E}(P_m)$ under the homeomorphism associated with ν. Also, it follows that P_m is a singleton iff given $\nu \in \mathscr{E}(P_m)$, the "Haar convolution" $\lambda * \nu$ is ergodic. Veech also obtains this result [14, Th. 11].

In summary, we have that the set $(\pi^*)^{-1}(E(Y, T)) \subset M(X, T)$ can be written as a disjoint union of the sets P_m, $m \in E(Y, T)$, and each set P_m is a Choquet simplex, that is, $\mu \in P_m$ is represented by a unique measure concentrated on $\mathscr{E}(P_m)$. In particular, if (Y, T) is uniquely ergodic then $M(X, T) = P_m$ and so every T-invariant measure has a unique ergodic decomposition. Finally, the system $\pi:(G;X,T)\to(Y,T)$ goes over naturally to the system $\pi^*:(G;E(X,T))\to E(Y,T)$ where the G action is by convolution and may not be a free action.

2.3. Generalisations. In this subsection we make some observations about how far the results in (2.2) will extend to the case of a general homomorphism $\Phi:(X,T)\to(Y,T)$.

If $m\in M(Y,T)$ then $P_m=\{\mu\in M(X,T):\Phi^*\mu=m\}$. However, it does not seem clear that one always assert that $P_m\neq\emptyset$, since the technique of integrating via Haar measure over fibres is no longer available. Nevertheless, conditions on T which allow one to say that $M(X,T)\neq\emptyset$ will sometimes also yield that $P_m\neq\emptyset$.

THEOREM 2.3.1. *Let T be amenable and let $\Phi:(X,T)\to(Y,T)$ be a homomorphism. If $m\in M(Y,T)$ then P_m is non-empty.*

PROOF. The amenability of T means that Stone-Čech compactification of T (with its discrete topology) admits a left invariant probability ω. Now βT acts as a semigroup of transformations (measurable but not necessarily continuous) on X by $p(x)=\lim t_nx$ if $p=\lim t_n$. The continuity of Φ, together with a similar action of βT on Y, yields that $\Phi p=p\Phi(p\in\beta T)$, and if $m\in M(Y,T)$, then $m\in M(Y,\beta T)$.

Let η be any (not necessarily T-invariant) probability measure on X with $\Phi^*\eta=m$; such a measure always exists by the Hahn-Banach theorem. Then define

$$\mu(f)=\int_X\left(\int_{\beta T}f(px)d\omega(p)\right)d\eta(x)\qquad f\in\mathscr{C}(x).$$

It is easy to show that $\mu\in M(X,T)$.

Some of the results we have for P_m in the case of a free G-extension carry over without alteration. However, we do lose the nice description of $\mathscr{E}(P_m)$. In the following we use the notation $P_m(\Phi)$ to denote the "lifted set" of m with respect to the homomorphism $\Phi:(X,T)\to(Y,T)$.

THEOREM 2.3.2. *Let $\Phi:(X,T)\to(Y,T)$ and let $m\in\mathscr{E}(Y,T)$ be such that $P_m(\Phi)$ is non-empty. Then*:

(1) *$P_m(\Phi)$ is a simplex*;

(2) $\mathscr{E}(P_m(\Phi))=E(X,T)\cap P_m(\Phi)$.

Now suppose that, in addition, we have a map $\psi:(Z,T)\rightarrow(X,T)$ and assume T is amenable. (Or more generally we can always "lift" measures.) Then:

(3) $\psi^*P_m(\Phi\circ\psi)=P_m(\Phi)$;

(4) $\psi^*(\mathscr{E}(P_m(\Phi\circ\psi)))=\mathscr{E}(P_m(\Phi))$.

PROOF. The proofs of (1) and (2) are the same as the proofs of the corresponding facts in Theorems 2.2.2 and 2.2.3.

(3) Clearly $\psi^*P_m(\Phi\circ\psi)\subset P_m(\Phi)$. Let $\omega\in P_m(\Phi)$. Then there exists a measure $\delta\in M(X,T)$ such that $\psi^*\delta=\omega$. But then $\Phi^*\psi^*\delta=\Phi^*\omega=m$, that is, $\delta\in P_m(\Phi\circ\psi)$. So $\psi^*P_m(\Phi\circ\psi)=P_m(\Phi)$.

(4) It follows from (2) and the fact that $\psi^*E(Z,T)=E(X,T)$ that $\psi^*\mathscr{E}(P_m(\Phi\circ\psi))\subset\psi^*\mathscr{E}(P_m(\Phi))$.

Now let $\omega\in\mathscr{E}(P_m(\Phi))$ and consider the set $P_\omega(\psi)$. If $\delta\in\mathscr{E}(P_\omega(\Phi))$, then by (2),

$$\delta\in E(X,T)\cap P_\omega(\psi)\subseteq E(X,T)\cap P_m(\Phi\circ\psi)=\mathscr{E}(P_m(\Phi\circ\psi)).$$

Hence, $\omega\in\psi^*E(P_m(\Phi\circ\psi))$.

The proof is complete.

As a special case we look at the splitting of a G-extension over a subgroup H. If H is normal, then all the homomorphisms involved are group extensions and so we gain no information. However, if H is not normal then we gain some information about the homomorphism $\pi_2:(X/H,T)\rightarrow(Y,T)$.

COROLLARY 2.3.3. *Let $\pi:(G;X,T)\rightarrow(Y,T)$ split into $\pi_1:(H;X,T)\rightarrow(X/H,T)$ and $\pi_2:(X/H,T)\rightarrow(Y,T)$, and let $m\in E(Y,T)$. Then $P_m(\pi_2)$ is a simplex and its extreme points form a compact set.*

PROOF. We know that we can "lift" measures via the group extensions π_1 and π. To see that we can lift measures via π_2 we simply note that if $m\in M(Y,T)$ then $\pi_1^*P_m(\pi)\subset P_m(\pi_2)$. Since $P_m(\pi)$ is non-empty, it follows that $P_m(\pi_2)$ is non-empty. We now apply Theorem 2.3.2 to obtain $P_m(\pi_2)=\pi^*P_m(\pi)$ and $\mathscr{E}(P_m(\pi_2))=\pi_1^*\mathscr{E}(P_m(\pi))$, and that $P_m(\pi_2)$ is a simplex. Since $\mathscr{E}(P_m(\pi))$ is compact by Corollary 2.2.4, it then follows that $\mathscr{E}(P_m(\pi_2))$ is compact.

3. Abelian group extensions

3.1. Abelian group extensions on metric spaces. Throughout this section, X and Y will denote compact metric spaces, G will denote a compact metric abelian group, and T will denote a locally compact separable group.

Our purpose in this section is to give a detailed analysis of the ergodic measures of a free G-extension $\pi:(G;X,T)\to(Y,T)$ with respect to the ergodic measures on (Y,T). Loosely speaking, we fix $m\in\mathscr{E}(Y,T)$ and associate with m the subgroup $H(\pi,m)$ (see (1.4)). Then we shall show that each $\nu\in\mathscr{E}(X,T)\cap P_m(\pi)$ is a "lift" of m via Haar measure on $H(\pi,m)$. To make this more precise, we make the following definition.

DEFINITION 3.1.1. Let H be a closed subgroup of G and split the G-extension π into the free H-extension $\pi_1:(H;X,T)\to(X/H,T)$ and the free G/H-extension $\pi_2:(G/H;X/H,T)\to(Y,T)$.

Let $m\in\mathscr{E}(Y,T)$ and let $\nu\in\mathscr{E}(X,T)\cap P_m(\pi)$. We say that ν is an H-*extension* of m if ν is the Haar lift (over H) of $\pi_1^*\nu$ and if the projection $\pi_2:(X/H,T,\pi_1^*\nu)\to(Y,T,m)$ is a measure isomorphism mod 0. That is, there exist *strictly* T-invariant Borel sets $B\subset X/H$, $C\subset Y$ with $\pi_1^*\nu(B)=1=m(C)$ such that π_2 is a Borel isomorphism from B onto C.

Our main result is that any $\nu\in\mathscr{E}(X,T)\cap P_m(\pi)$ is an $H(\pi,m)$-extension of m.

We first consider the extreme cases when $H(\pi,m)=G$ and $H(\pi,m)=\{e\}$.

THEOREM 3.1.2. *Let (X,T) be a free G-extension of (Y,T) and let $m\in\mathscr{E}(Y,T)$. If $H(\pi,m)=G$, then the Haar lift $\tilde{m}$ of m belongs to $\mathscr{E}(X,T)$ and hence is the only measure in P_m.*

PROOF. The condition $H(\pi,m)=G$ means that if the equation

$$\frac{f\circ\pi\circ t}{f\circ\pi}=\frac{f_\gamma\circ t}{f_\gamma}[\pi^{-1}m]\qquad(t\in T)$$

has a solution $f\in\mathscr{B}(Y,K)$, then $\gamma\equiv 1$. The proof of this theorem is a straightforward duplication of the proof of Theorem 3 in [8]. Notice that Parry only considers an action of the integers ($T=\mathbf{Z}$) but his proof makes no use of this fact. The last remark follows from Corollary 1.5.6.

THEOREM 3.1.3. *Let (X,T) be a free G-extension of (Y,T) and let $m\in$*

$E(Y,T)$. If $H(\pi,m)=\{e\}$ and $\nu\in\mathscr{E}(X,T)$ with $\pi^*\nu=m$, then $\pi:(X,T,\nu)\to(Y,T,m)$ is a measure isomorphism mod 0.

PROOF. For each $\gamma\in\chi(G)=\Gamma(\pi,m)$ (see (1.4)), choose a function f_γ of type γ such that $f_\gamma\circ t=f_\gamma[\nu]$ $(t\in T)$.

Since ν is ergodic, f_γ is constant $[\nu]$, say $f_\gamma=\alpha_\gamma[\nu]$. Let $B=\{x\in X:f_\gamma(x)=\alpha_\gamma$ for all $\gamma\in\chi(G)\}$. Since $\chi(G)$ is countable, B is a Borel set with $\nu(B)=1$. Since each f_γ is T-invariant $[\nu]$, it follows that B is T-invariant $[\nu]$. Now we replace B by $B_0=\{x:xt\in B$ for ζ-almost all $t\in T\}$ where ζ denotes a left invariant Haar measure on T. By Lemma 1.7.1, B_0 is a *strictly* T-invariant Borel set and $\nu(B_0\Delta B)=0$, that is, B_0 also satisfies $\nu(B_0)=1$.

We now show that B_0 intersects each orbit in at most one point. If $x,gx\in B_0$, then there exists a $t\in T$ such that $tx,tgx\in B_0$. But then for each $\gamma\in\chi(G)$, $f_\gamma(tx)=f_\gamma(tgx)=\gamma(g)f_\gamma(tx)$. Thus $\gamma(g)=1$ for each $\gamma\in\chi(G)$, and so $g=e$.

The map $\pi:B_0\to Y$ is a one-to-one measurable map and since X and Y are metric spaces we can use Kuratowski's theorem [9, p. 22] to say that $\pi(B_0)$ is a Borel set and $\pi:B_0\to\pi B_0$ is a Borel isomorphism. Since $\pi\circ t=t\circ\pi$, it follows that πB_0 is a strictly T-invariant set, and the fact that $\pi^*\nu=m$ implies that $m(\pi B_0)=\nu(\pi^{-1}\pi B_0)=\nu(\pi^{-1}\pi B_0\cap B_0)=\nu(B_0)=1$. This proves the result.

Finally we combine Theorems 3.1.2 and 3.1.3 to deal with the general case.

THEOREM 3.1.4. *Let (X,T) be a free G-extension of (Y,T) and let $m\in\mathscr{E}(Y,T)$. Then every $\nu\in\mathscr{E}(X,T)\cap P_m(\pi)$ is an $H(\pi,m)$-extension of m.*

PROOF. For brevity in this proof we denote $H(\pi,m)$ by H. Consider the splitting of $\pi:(G;X,T)\to(Y,T)$ by the subgroup H. That is, we have a free H-extension $\pi_1:(H;X,T)\to(X/H,T)$ and a free G/H-extension $\pi_2:(G/H;X/H,T)\to(Y,T)$. Let $m\in E(T,T)$ and let $\nu\in E(X,T)\cap P_m$. In order to prove the theorem we will show that $\Gamma(\pi_2,m)=\chi(G/H)$ and $\Gamma(\pi_1,\pi_1^*\nu)=\{1\}$, and then apply Theorems 3.1.3 and 3.1.2 to obtain the desired result.

First we note that since $H=\operatorname{ann}\Gamma(\pi,m)$, we can identify $\chi(G/H)$ with $\Gamma(\pi,m)$ in the sense that if $\gamma\in\Gamma(\pi,m)$, then γ is constant on H-cosets and uniquely defines a $\gamma'\in\chi(G/H)$. It is well known that all of $\chi(G/H)$ can be obtained in this way.

We first show $\Gamma(\pi_2,m)=\Gamma(\pi,m)$. Choose $\gamma\in\Gamma(\pi,m)$ and a function f_γ of type γ satisfying $f_\gamma\circ t=f_\gamma$ $(t\in T)$. Since $f_\gamma(hx)=\gamma(h)f_\gamma(x)=f_\gamma(x)$ for all $h\in H$, it follows that f_γ is constant on H-orbits and so determines a function $f_{\gamma'}$

on X/H which is of type γ' and satisfies $f_{\gamma'} \circ t = f_{\gamma'}[\pi_2^{-1}m]$ $(t \in T)$. Therefore, $\gamma' \in \Gamma(\pi_2, m)$, and since the other inclusion is obvious, we obtain $\Gamma(\pi_2, m) = \Gamma(\pi, m)$. We now apply Theorem 3.1.3 to the free G/H-extension $\pi_2 : (G/H; X/H, T) \to (Y, T)$. Since $\pi_1^* \nu \in E(X/H, T)$ and $\pi_2^* \pi_1^* \nu = \pi^* \nu = m$ there is a strictly T-invariant Borel set $B \subset X/H$ of full $\pi_1^* \nu$-measure which intersects each G/H-orbit in at most one point. This means that any Borel function f on X/H is equal $[\pi_1^* \nu]$ to a Borel function f' which is constant on G/H orbits, namely, the Borel function f' defined by

$$f'(x) = \begin{cases} f(x_0) & \text{if } \{x_0\} = G/Hx \cap B \\ 1 & \text{if } \varnothing = G/Hx \cap B. \end{cases}$$

We now make use of this new function f' to show that $\Gamma(\pi_1, \pi_1^* \nu) = \{1\}$. If $\gamma' \in \chi(H)$, then it is known that one can choose $\gamma \in \chi(G)$ such that $\gamma|_H = \gamma'$. If f_γ is a function of type γ for the extension π, then f_γ is also a function of type $\gamma' = \gamma|_H$ for the extension π_1. We then study the functional equation

$$\frac{f \circ \pi_1 \circ t}{f \circ \pi_1} = \frac{f_{\gamma'} \circ t}{f_{\gamma'}} [\pi_1^{-1} \pi_1^* \nu] \quad (t \in T)$$

where $f \in \mathcal{B}(X/H, K)$. As we remarked in (1.3), if this equation has a solution f for one function $f_{\gamma'}$ of type γ' for π_1, then it has a solution for every function $f_{\gamma'}$ of type γ' for π_1. Thus, we need only consider $f_{\gamma'}$ where $\gamma|_H = \gamma'$, and f_γ as above.

Suppose
$$\frac{f \circ \pi_1 \circ t}{f \circ \pi_1} = \frac{f_\gamma \circ t}{f_\gamma \circ t} [\pi_1^{-1} \pi_1^* \nu] \quad (t \in T)$$

where $f \in \mathcal{B}(X/H, K)$. Choose a function $f' \in \mathcal{B}(X/H, K)$ which is constant on G/H-orbits and such that $f' = f[\pi_1^* \nu]$ as in the previous paragraph. Then f' defines a function $f'' \in \mathcal{B}(Y, K)$ such that $f'' \circ \pi_2 = f'$. Putting this into the equation we obtain

$$\frac{f'' \circ \pi_2 \circ \pi_1 \circ t}{f'' \circ \pi_2 \circ \pi_1} = \frac{f_\gamma \circ t}{f_\gamma} [\pi_1^{-1} \pi_1^* \nu] \quad (t \in T).$$

This equation is now measurable with respect to $\pi^{-1}\mathcal{B}(Y)$ and $\pi_2 \circ \pi_1 = \pi$, so we obtain

$$\frac{f''\circ\pi\circ t}{f''\circ\pi}=\frac{f_\gamma\circ t}{f_\gamma}\ [\pi^{-1}m]\quad (t\in T).$$

In other words, $\gamma\in\Gamma(\pi,m)=\operatorname{ann}H$, that is, $\gamma|_H=1$. This shows that $\Gamma(\pi_1,\pi_1^*\nu)=\{1\}$ and hence by Theorem 3.1.2, ν is the Haar lift (over H) of $\pi_1^*\nu$. The proof is complete.

As a corollary to this result we give a precise version of a theorem of Furstenburg [5, p. 591].

COROLLARY 3.1.5. *If (X,T) is a free circle extension of (Y,T), then every $\nu\in\mathscr{E}(X,T)$ is either the Haar lift of $\pi^*\nu$ or is a finite subgroup extension of $\pi^*\nu$. The subgroup depends only on $\pi^*\nu$.*

PROOF. The only closed subgroups of the circle are either finite or the whole circle.

In Section 5 we give a collection of examples of circle extensions in which each of the possible subgroups mentioned in Corollary 3.1.5 occurs.

Finally, we note that in our situation, two subgroups arise naturally: $H(\pi,m)$ and $H_\nu=\{g\in G: g\nu=\nu\}$, if $\nu\in\mathscr{E}(P_m)$ (note that since G is abelian, $H_\nu=H_{g\nu}$). Following an observation of J. Johnson, we can show that these subgroups are identical.

COROLLARY 3.1.6. $H(\pi,m)=H_\nu$.

PROOF. Suppose $g\in H_\nu$. Then $g\nu=\nu$. If $\gamma\in\chi(G)$ satisfies $f_\gamma\circ t=f_\gamma[\pi^{-1}m]$ $(t\in T)$, then $f_\gamma\circ t=f_\gamma[\nu]$ $(t\in T)$. Since ν is ergodic, $f_\gamma=C[\nu]$, C a constant, $C\neq 0$. But then $C=\int f_\gamma d\nu=\int f_\gamma dg\nu=\gamma(g^{-1})\int f_\gamma d\nu$. Hence, $\gamma(g^{-1})=1$, and $\gamma(g)=1$. Thus $g\in H(\pi,m)$.

Suppose $g\in H(\pi,m)$. Pick $\nu\in\mathscr{E}(P_m)$. Now $\pi_1^*\nu=\pi_1^*g\nu$, using the notation of Theorem 3.1.4. Since ν is obtained by the Haar lift over $\pi_1^*\nu$, and $g\nu$ by the Haar lift over $\pi_1^*g\nu$, this means that $\nu=g\nu$, and $g\in H_\nu$, as desired.

3.2. Remarks on non-metric case. In this subsection we mention briefly some of the problems involved in removing the metric assumptions on X and Y. Theorem 3.1.2 is in fact valid for non-metric X, Y. The construction in Theorem 3.1.3 can still be carried out to obtain a strictly T-invariant Borel set which intersects orbits in at most one point. However, we know of no non-metric version of Kuratowski's theorem which would then allow us to say that π restricted to B was an isomorphism. We could, with a little extra work to ensure that $\pi(B)$ is Borel, assert that π is a one-to-one homomorphism mod 0. This gives a somewhat weaker version of Theorem 3.1.3. Our proof of Theorem 3.1.4 now breaks down because we can no longer show $\Gamma(\pi_1, \pi_1^*\nu) = \{1\}$. Our proof involves replacement of Borel functions on X/H by Borel functions which are constant on G/H-orbits, and we need Borel measurability of the inverse of π_2 restricted to $\pi(B)$ in order to do this.

4. Inverse limit systems

In this section we will prove some general results on inverse limit systems. Unless otherwise stated spaces, X, X_α, etc., are compact Hausdorf, and T is arbitrary.

DEFINITION 4.1. Let A be a directed set. We say that a collection $(X_\alpha, T, \nu_\alpha)$, $\alpha \in A$, forms an *inverse limit system* if:

(i) $\nu_\alpha \in M(X_\alpha, T)$, for each $\alpha \in A$;

(ii) if $\alpha > \beta$ there exists a homomorphism $\pi_{\beta\alpha} : (X_\alpha, T) \to (X_\beta, T)$ such that $\pi^*_{\beta\alpha}\nu_\alpha = \nu_\beta$;

(iii) if $\alpha > \beta > \gamma$ then $\pi_{\gamma\alpha} = \pi_{\gamma\beta} \circ \pi_{\beta\alpha}$.

We denote the inverse limit of the transformation groups (X_α, T) by (X, T) and denote by $\pi_\alpha : (X, T) \to (X_\alpha, T)$ the natural homomorphisms. Note that if $\alpha > \beta$ then $\pi_\beta = \pi_{\beta\alpha}\pi_\alpha$.

THEOREM 4.2. *Let $(X_\alpha, T, \nu_\alpha)$, $\alpha \in A$, form an inverse limit system. Then there exists a unique measure $\nu \in M(X, T)$ such that $\pi^*_\alpha\nu = \nu_\alpha$ for each $\alpha \in A$. We say that (X, T, ν) is the inverse limit of the system $(X_\alpha, T, \nu_\alpha)$.*

PROOF. This is a standard proof, using the Stone-Weierstrasse theorem to construct a linear functional on $C(X)$.

If to each transformation group (X_α, T), we associate the compact Hausdorf space $M(X_\alpha, T)$ then we obtain an inverse limit system $M(X_\alpha, T)$, $\alpha \in A$, with the maps $\pi^*_{\beta\alpha} : M(X_\alpha, T) \to M(X_\beta, T)$ when $\alpha > \beta$.

THEOREM 4.3. *The compact Hausdorf space $M(X, T)$ is homeomorphic to the inverse limit of the system $M(X_\alpha, T)$ $(\alpha \in A)$.*

PROOF. This follows easily from the definition of the two topologies involved.

In our next theorem we show that if each $(X_\alpha, T, \nu_\alpha)$ is an ergodic system then the inverse limit (X, T, ν) is ergodic.

THEOREM 4.4. *Let* $(X_\alpha, T, \nu_\alpha)$ *be an inverse limit system with inverse limit* (X, T, ν). *Then* $\nu_\alpha \in E(X_\alpha, T)$ *for each* $\alpha \in A$ *if and only if* $\nu \in E(X, T)$.

PROOF. It is clear that $\nu \in E(X, T)$ implies $\nu_\alpha \in E(X_\alpha, T)$ for each $\alpha \in A$. Now let us suppose $\nu_\alpha \in E(X, T)$ for each α, and suppose $\nu \notin E(X, T)$. Then $\nu = \beta\mu + \delta\omega$, with $\mu \neq \omega$, $\beta, \delta > 0$, and $\beta + \delta = 1$. It follows that for every α, $\nu_\alpha = \pi_\alpha^* \nu = \beta\pi_\alpha^*\mu + \delta\pi_\alpha^*\omega$. Since ν_α is ergodic, $\nu_\alpha = \pi_\alpha^*\mu = \pi_\alpha^*\omega$. Hence, $\mu = \omega = \nu$, a contradiction, and thus $\nu \in E(X, T)$. This completes the proof.

COROLLARY 4.5. *The subspace* $E(X, T)$ *can be identified with the inverse limit of the subspaces* $E(X_\alpha, T)$.

PROOF. It is direct to verify that the relative topology of $E(X, T)$ coincides with the inverse limit of the relative topologies of $E(X_\alpha, T)$, $\alpha \in A$.

COROLLARY 4.6. *The inverse limit* (X, T, ν) *of an inverse system* $(X_\alpha, T, \nu_\alpha)$, $\alpha \in A$, *of uniquely ergodic transformation groups is uniquely ergodic.*

We now look at some special types of inverse limits which are defined using free G-extensions. These are a subclass of the quasi-isometric extensions considered by Furstenburg in his study of distal flows [6] in the case when X is metric.

DEFINITION 4.7. Let (X, T) be a transformation group and (Y, T) a subflow. Suppose there is an ordinal η such that for each $\xi \leqq \eta$, there is a subflow (X_ξ, T) such that:

(i) $(Y, T) = (X_0, T)$, $(X, T) = (X_\eta, T)$.

(ii) If $\xi < \xi'$, there is a map $\pi_{\xi\xi'} : (X_{\xi'}, T) \rightarrow (X_\xi, T)$ and the system (X_ξ, T), $\xi \leqq \eta$, together with these maps forms an inverse system.

(iii) For each $\xi < \eta$, $(X_{\xi+1}, T)$ is a free G_ξ-extension of (X_ξ, T) for some G_ξ.

(iv) If $\xi \leqq \eta$ is a limit ordinal, then (X_ξ, T) is the inverse limit of $(X_{\xi'}, T)$, $\xi' < \xi$.

We will say that (X, T) is a *free* $\{G_\xi\}$-*extension* of (Y, T).

We can now combine our final remark of Section 2 with Corollary 4.5 to obtain:

COROLLARY 4.8. *If* (X, T) *is a free* $\{G_\xi\}$-*extension of* (Y, T), *then* $E(X, T)$ *is a* $\{G_\xi\}$-*extension of* $E(Y, T)$. (*Regarding* $E(X, T)$, *etc., simply as sets.*)

If we restrict ourselves to the situation of Section 3, then we can say a little more.

COROLLARY 4.9. *If X is a compact metric space, T a locally compact separable group and G_ξ a compact metric abelian group for each $\xi \leqq \eta$, then each measure $\nu \in E(X,T)$ is associated with a sequence $(\nu_\xi \mid \xi \leqq \eta)$ of measures $\nu_\xi \in E(X_\xi, T)$ such that $\nu_{\xi+1}$ is an $H(\pi_{\xi,\xi+1}, \nu_\xi)$-extension of ν_ξ (using the notation of Section 3), and $\nu_\eta = \nu$.*

In particular, if (Y,T) is uniquely ergodic, and $\nu_0 \in E(Y,T)$, then every $\nu \in E(X,T)$ is "determined" by ν_0 and $(H(\pi_{\xi,\xi+1}, \nu_\xi) \mid \xi \leqq \eta)$.

5. Examples

5.1. Circle extensions. In this subsection we take an example due to Furstenberg [5, p. 585] and examine its ergodic measures. We then modify this example by using translations by various points of the circle and show how to obtain all the possibilities mentioned in Corollary 3.1.5.

Let $Y = K$ and let $X = K \times K$ with the natural K-action on the second factor, $g(x,y) = (x, gy)$. We are going to study certain Z-actions on X and Y, that is, we study homeomorphisms. Let $\Phi: K \to K$ be defined by $\Phi x = \alpha x$, $\alpha \in K$, and let $\psi: K \times K \to K \times K$ be defined by $\psi(x,y) = (\alpha x, g(x)y)$ where $g: K \to K$. Letting π be the first projection, $\pi(x,y) = x$, then π: $(K ; X, \psi) \to (Y, \Phi)$ is a free K-extension.

The character group of the circle is isomorphic to Z and it is easily verified that if $m \in \mathscr{E}(Y,\Phi)$ then

$$\Gamma(\pi, m) = \left\{ n : \frac{g^n(\alpha x)}{g^n(x)} = \frac{f(\alpha x)}{f(x)} [m] \text{ has a solution } f \in \mathscr{B}(K,K) \right\}$$

Notice that $\Gamma = n_0 Z$ where n_0 is the smallest positive element of Γ and then $H(\pi, m) =$ group of n_0th roots of unity.

In [5] Furstenberg constructs $\alpha = e^{2\pi i \beta}$, β irrational (hence (Y,Φ) is strictly ergodic), and a Borel function $\hat{R}: K \to K$ such that $\hat{R}(\alpha x)/\hat{R}(x)$ is continuous and $\hat{R}(\alpha x)/\hat{R}(x) = f(\alpha x)/f(x)$ has no continuous solution f. Let ψ be the homeomorphism defined by $\psi(x,y) = (\alpha x, (\hat{R}(\alpha x)/\hat{R}(x))y)$. Furstenberg asserts that (X, ψ) is minimal and not strictly ergodic, and in [2, pp. 14–20] Effros and Hahn give a detailed proof that this homeomorphism is minimal, and discuss its ergodic measures. They prove that each ergodic measure is a one-point extension of Lebesgue measure on Y by constructing specific isomorphisms. We will indicate how to obtain this result from our work in Section 3. We then go on to discuss the ergodic measures for some related homeomorphisms. Below, we shall assume $R: K \to K$ is Borel and $R(\alpha x)/R(x)$ is continuous.

PROPOSITION 5.1.1. *If* $\psi(x,y)=(\alpha x,(R(\alpha x)/R(x))y)$, *then each ergodic measure for* ψ *is a* $\{1\}$*-extension of Lebesgue measure on* Y.

PROOF. The proof involves showing that $\Gamma(\pi,m)=\mathbf{Z}$ which is a trivial verification.

PROPOSITION 5.1.2. *If* $\psi_n(x,y)=(\alpha x,(R(\alpha x)/R(x))\omega_n y)$, *where* ω_n *is a primitive* nth *root of unity, then each ergodic measure for* ψ_n *is an* $\{1,\omega_n,\cdots,\omega_n^{n-1}\}$*-extension of Lebesgue measure on* Y.

PROOF. $(R^k(\alpha x)/R^k(x))\omega_n^k=f(\alpha x)/f(x)$ has a solution if and only if ω_n^k is an eigenvalue of Φ. Since α is not a root of unity, this occurs if and only if $k=n$. Therefore $\Gamma(\pi,m)=n\mathbf{Z}$ and the annihilator of $\Gamma(\pi,m)$ is the subgroup $\{1,\omega_n,\cdots,\omega_n^{n-1}\}$.

These examples together with Furstenberg's function show that one can obtain all finite subgroups of the circle as $H(\pi,m)$'s even in the minimal case.

Finally we obtain the whole circle, that is, we construct uniquely ergodic systems (X,T).

PROPOSITION 5.1.3. *If* $\psi_\beta(x,y)=\alpha x,(R(\alpha x)/R(x))\beta y)$, *and if* $\beta^k\notin\{\alpha^n:n\in\mathbf{Z}\}$ *for every* $k\neq 0$, *then* ψ_β *is uniquely ergodic.*

PROOF. The equation $(R^k(\alpha x)/R^k(x))\beta^k=f(\alpha x)/f(x)$ has a solution if and only if β^k is an eigenvalue of Φ. Our choice of β precludes this possibility for $k\neq 0$, and therefore there is a solution if and only if $k=0$. Therefore $\Gamma(\pi,m)=\{0\}$, and so ψ_β has only one ergodic measure. This completes the proof.

5.2 Groups with monothetic subgroups. Given an arbitrary compact metric abelian group G the problem of which closed subgroups of G can appear as an $H(\pi,m)$ in a discrete flow seems to be quite difficult. We have seen in (5.1) that for the circle we can obtain all closed subgroups. As a slight generalization of this, we note the following types of free group extensions.

Let H be a monothetic subgroup of G (that is, there is an element $h_0\in H$ such that H is equal to the closure of the cyclic group generated by h_0). Define a homeomorphism $\psi: Y\times G\to Y\times G$ by $\psi(y,g)=(\Phi y,gh_0)$, where Φ is a specified homeomorphism of Y, and $\pi:(G;Y\times G,\psi)\to(Y,\Phi)$ is the projection. Note that if H is connected, then H must be monothetic.

PROPOSITION 5.2.1. *If* m *is a measure on* Y *for which* (Y,Φ,m) *is weak mixing, then* $H(\pi,m)=H$.

PROOF. Let $\gamma \in \chi(G)$. Then $\gamma \in \Gamma(\pi, m)$ if and only if the equation $\frac{f(\Phi y)}{f(y)} = \gamma(h)$ $[m]$ $(h \in H)$ has a solution for $f \in \mathcal{B}(Y, K)$. Since Φ is weak mixing with respect to m this equation has a solution if and only if $\gamma(h) = 1$ $(h \in H)$. But then $\gamma \equiv 1$ on H, and so $\gamma \in \operatorname{ann} H$. Thus, $\Gamma(\pi, m) = \operatorname{ann} H$, and $H(\pi, m) = H$.

If one is willing to have any arbitrary group act, then there is a trivial solution to the above problem: consider the system $(G; G, H)$ (i.e., $T = H$). Then the above analysis (with Y a point, and m point mass) shows that $H(\pi, m) = H$.

5.3 Connected group extensions. It was pointed out in [2] that in the case of a circle extension of a uniquely ergodic homeomorphism, the number of ergodic measures is either uncountable or one. Using our results of Section 2, we can make a similar observation about connected group extensions with arbitrary acting group T.

PROPOSITION 5.3.1. *Let (X, T) be a free G-extension of (Y, T), with G connected, and let $m \in E(Y, T)$. Then either $\mathcal{E}(P_m)$ is uncountable, or $\mathcal{E}(P_m)$ is a singleton (notation as in Section 2).*

PROOF. By Theorem 2.2.5, $\mathcal{E}(P_m)$ is homeomorphic to G/H for some closed subgroup H of G. Since G is connected, it follows that G/H is connected, and hence is either uncountable or a singleton.

5.4 Entropy of group extensions. In this subsection the acting group T will be the integers, that is, we are concerned with homeomorphisms. Our purpose here is to link the results we have with some known results on entropy. For a discussion of entropy, both topological and measure-theoretic, we refer the reader to the lecture notes [15]. We use the notation $h(\Phi)$ for the topological entropy of Φ and the notation $h_m(\Phi)$ for the measure-theoretic entropy of Φ with respect to a measure m. The space X will be assumed to be a metric space. This insures that we are working with Lebesgue spaces when we discuss measure-theoretic entropy.

PROPOSITION 5.4.1. *Let (X, Φ) be a free G-extension of (Y, ψ), and let $m \in E(Y, \psi)$. Then for all $\mu, \mu_1 \in P_m(X, \Phi)$, we have $h_\mu(\Phi) = h_{\mu_1}(\Phi)$.*

PROOF. Fix $\nu_1, \nu_2 \in \mathcal{E}(P_m)$. Then there exists a $g \in G$ such that $g\nu_1 = \nu_2$. The homeomorphism g gives a measure isomorphism of the systems (X, ν_1, Φ) and (X, ν_2, Φ), therefore $h_{\nu_1}(\Phi) = h_{\nu_2}(\Phi)$. This proves the result for the ergodic measures in $P_m(X, \Phi)$. The general result now follows from the ergodic decompositions of measure-theoretic entropy, completing the proof.

PROPOSITION 5.4.2. *With the notation of Proposition* 5.4.1, *for all* $\mu \in P_m(X,\Phi)$, *we have* $h_\mu(\Phi) = h_m(\psi)$.

PROOF. By Proposition 5.4.1, it suffices to show that $h_{\tilde{m}}(\Phi) = h_m(\psi)$, where $\tilde{m}$ is the Haar lift of m. But this has been proved by R. K. Thomas [11, p. 119].

We say that a measure μ has *maximal entropy* for Φ if $h_\mu(\Phi) = h(\Phi)$.

PROPOSITION 5.4.3. *Suppose that m has maximal entropy for* ψ. *Then for all* $\mu \in P_m(X,\Phi)$, μ *has maximal entropy for* Φ.

PROOF. H. B. Keynes [7] has shown that $h(\Phi) = h(\psi)$ and Proposition 5.4.2 shows that for all $\mu \in P_m(X,\Phi)$ we have $h_\mu(\Phi) = h_m(\psi)$. So if $h(\psi) = h_m(\psi)$, then $h(\Phi) = h_\mu(\Phi)$ for all $\mu \in P_m(X,\Phi)$.

PROPOSITION 5.4.4. *Suppose that* (Y,ψ) *has a unique measure m with maximal entropy (which must then be ergodic). Then* (X,Φ) *has a unique measure with maximal entropy if and only if* $\tilde{m}$ *is ergodic.*

PROOF. (X,Φ) has a unique measure with maximal entropy if and only if $P_m(X,\Phi)$ is a singleton. But this is equivalent to $\tilde{m}$ being ergodic (see Corollary 1.5.6 and Theorem 2.2.5).

6. Minimal sets in G-extensions

In this section, G denotes a compact abelian group. Following Parry [9], we make the following definition.

DEFINITION 6.1. We say that $\pi:(G;X,T)\rightarrow(Y,T)$ is a *free (topologically) simple* G-extension if there exist continuous functions of type γ for each $\gamma \in \chi(G)$.

Throughout this section, f_γ will denote a continuous function of type γ.

We will indicate a result analogous to Theorem 3.1.4 for minimal subsets of simple extensions rather than for ergodic measures. Similar but somewhat weaker results are known in general [4, Prop. 6.4 and Remark 6.5]. We make no assumptions on (X,T) other than X being compact Hausdorff.

Let M be a minimal subset of (Y,T). We define $\Gamma(\pi,M) = \{\gamma : f_\gamma(tx) = f_\gamma(x)$ $(t \in T,\ x \in \pi^{-1}M)$ for some $f_\gamma\}$. Then $\Gamma(\pi,M)$ is a subgroup of $\chi(G)$ and we denote its annihilator by $H(\pi,M)$.

DEFINITION 6.2. *Let H be a closed subgroup of G. A minimal subset N of* (X,T) *such that* $N \subset \pi^{-1}M$ *is said to be an H-extension of M if in the splitting*

$\pi_1: (H; X, T) \rightarrow (X/H, T)$, $\pi_2: (G/H; X/H, T) \rightarrow (Y, T)$ *the homomorphism* $\pi_2: (\pi_1(N), T) \rightarrow (M, T)$ *is a isomorphism and* $N = \pi_1^{-1}\pi_1(N)$.

THEOREM 6.3. *Suppose M is a minimal subset of* (Y, T) *and N is a minimal subset of* (X, T) *with* $N \subset \pi^{-1}M$. *Then N is an* $H(\pi, M)$*-extension of M.*

We list a few remarks which will enable the interested reader to construct a proof.

1. The result analogous to Theorem 3.1.2 is based on Theorem 1 in Parry [9, p. 98]. The extension of his work from an integer action to an arbitrary action involves no extra work. To extend his work from a compact metric X to a compact Hausdorf X involves replacing the sequential convergence in Parry's argument by net convergence.

2. The argument for a result analogous to Theorem 3.1.3 is easy. If $\Gamma(\pi, M) = \chi(G)$, then on a minimal subset $N \subset \pi^{-1}M$ each f_γ satisfying $f_\gamma \circ t = f_\gamma$ is a constant. Taking the intersection of the sets of points where $f_\gamma = \alpha_\gamma$ we obtain a closed invariant set which contains N and intersects each orbit in $\pi^{-1}M$ in exactly one point. It is easy to see that this set is in fact N. The projection π is one-to-one and continuous, hence an isomorphism.

3. The proof of Theorem 6.3 can now be completed following the lines of Theorem 3.1.4.

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